ISRAEL JOURNAL OF MATHEMATICS, Vol. 43, No. 3, 1982

UPPER AND LOWER BOUNDS OF THE VALENCE-FUNCTIONAL

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ABSTRACT

For the non-negative integer g let (M, g) denote the closed orientable 2-dimensional manifold of genus g. K-realizations $\mathcal{P}$ of (M, g) are geometric cell-complexes in $\mathbb{R}^3$ with convex facets such that set $(\mathcal{P})$ is homeomorphic to M. For K-realizations $\mathcal{P}$ of (M, g) and vertices v of $\mathcal{P}$, $\mathrm{val}(v, \mathcal{P})$ denotes the number of edges of $\mathcal{P}$ incident with v and the weighted vertex-number $\Sigma(\mathrm{val}(v, \mathcal{P}) - 3)$ taken over all vertices of $\mathcal{P}$ is called valence-value $v(\mathcal{P})$ of $\mathcal{P}$. The valence-functional V, which is important for the determination of all possible f-vectors of K-realisations of (M, g), in connection with Eberhard's problem etc., is defined by $V(g) := \min[v(\mathcal{P}) \mid \mathcal{P}$ is a K-realization of $(M, g)]$. The aim of the note is to prove the inequality $2g + 1 \leqq V(g) \leqq 3g + 3$ for every positive integer g.

1. Introduction

For the non-negative integer g let (M, g) denote the closed orientable 2-dimensional manifold of genus g. K-realizations $\mathcal{P}$ of (M, g) are geometric cell-complexes in $\mathbb{R}^3$ with convex facets such that $\mathrm{set}(\mathcal{P})$ is homeomorphic to M. The 2-dimensional sphere can be realized as a convex polytope in $\mathbb{R}^3$, where every vertex is convex in the sense that at least one of the two components into which the set of the cell-complex divides a sufficiently small ball centered at the vertex is convex. This property characterizes genus 0.

In [2] and [3] the minimal number $H(g)$ of non-convex vertices of K-realizations of (M, g) is considered. Barnette proves $H(g) \leqq 7$ and in [3] the existence of K-realization of (M, g) with at most six non-convex vertices is shown for every positive integer g.

These results show that it is in fact possible to characterize the topological property of M to be a sphere by means of the geometric functional H, however, $H(g)$ is not the appropriate subject to get extensive characterizations for $g \neq 0$.

Received September 28, 1980 and in revised form May 13, 1981

Therefore we consider a functional which is more suitable for this purpose and which is furthermore important for several problems such as the determination of all possible f-vectors of K-realizations of (M,g) or in connection with Eberhard's problem.

Let $\mathcal{P}$ be a K-realization of (M,g). For vertices v of $\mathcal{P}$ let $\mathrm{val}(v,\mathcal{P})$ denote the number of edges of $\mathcal{P}$ being incident with v.

Then we have $\mathrm{val}(v,\mathcal{P}) \geqq 3$ and we define the valence-value $v(\mathcal{P})$ of $\mathcal{P}$ to be the weighted vertex-number $\Sigma(\mathrm{val}(v,\mathcal{P})-3)$ taken over all vertices of $\mathcal{P}$.

Since we want to obtain results which characterize the genus g rather than some special K-realizations of (M,g) we define the valence-functional $V(g)$ to be the minimum of all $v(\mathcal{P})$, i.e.

$$V(g) := \min[v(\mathcal{P}) \mid \mathcal{P} \text{ is a } K\text{-realization of } (M,g)].$$

For given g the valence-functional measures the "minimal valence-distance" between K-realizations of (M,g) and simple realizations, where all the vertices are 3-valent. Since $H(g) \leqq V(g)$ the sphere is characterized by the property $V(g)=0$.

In this note we prove that there is a "linear correspondence" between g and $V(g)$: for every positive integer the inequality $2g+1 \leqq V(g) \leqq 3g+3$ holds.

2. Preliminary definitions and results

Our notation mainly follows the terminology of [5].

Let $\mathcal{A}$ be a 2-dimensional cell-complex in $\mathbb{R}^3$ with convex facets. For $i=0,1,2$ $F_i(\mathcal{A})$ denotes the set of i-dimensional faces of $\mathcal{A}$ and $f_i(\mathcal{A})$ its cardinality.

Suppose $B := \mathrm{bd}(\mathrm{conv}(\mathrm{set}(\mathcal{A})))$ and let $\mathcal{B}$ be a K-realization of $(M,0)$ with set $(\mathcal{B}) = B$.

$\mathcal{B}$ is called associated with $\mathcal{A}$, if $F_0(\mathcal{B}) \subset F_0(\mathcal{A})$ and if every face of $\mathcal{P}$ belonging to B is a face of $\mathcal{B}$ too. $\mathbf{S}^2$ denotes the 2-dimensional unit sphere in $\mathbb{R}^3$ and for $z \in \mathbf{S}^2$, $p \in \mathbb{R}^3$, $H(z,p)$ indicates the hyperplane $\{x \mid x \in \mathbb{R}^3 \wedge (z,x) = (z,p)\}$, $H^+(z,p)$, $H^-(z,p)$ the closed halfspace $\{x \mid x \in \mathbb{R}^3 \wedge (z,x) \geqq (z,p)\}$, $\{x \mid x \in \mathbb{R}^3 \wedge (z,x) \leqq (z,p)\}$, respectively.

Now we introduce the "index of a vertex" which has been considered by Banchoff in [1] in a slightly different way.

Suppose $\mathcal{P}$ to be a K-realization of (M,g) and let $\mathbf{S}(\mathcal{P})$ be the set of all directions z of $\mathbf{S}^2$ for which all the hyperplanes $H(z,p)$ meet at most one vertex of $\mathcal{P}$. For every $v \in F_0(\mathcal{P})$ and $z \in \mathbf{S}(\mathcal{P})$ let $\mathcal{G}(P;z,v)$ denote the set of all facets of $\mathrm{st}(v,\mathcal{P})$ which meet $H(z,v)$ in more than one point. Then we define

$$w_{\mathscr{P}}(z, v) := \pi(2\text{-card}(\mathscr{G}(\mathscr{P}; z, v))).$$

Let $\mathscr{F}_1$ be the set of edges of $\mathscr{P}$ incident with v and contained in $H^+(z, v)$ and $\mathscr{F}_2$ the set of facets of $\mathrm{st}(v, \mathscr{P})$ contained in $H^+(z, v)$. Since $\mathrm{link}(v, \mathscr{P})$ is topologically a circle we have

$$2\,\mathrm{card}(\mathscr{F}_1) = \mathrm{card}(\mathscr{G}(\mathscr{P}; z, v)) + 2\,\mathrm{card}(\mathscr{F}_2),$$

thus $\mathrm{card}(\mathscr{G}(\mathscr{P}; z, v))$ is even and

$$w_{\mathscr{P}}(z, v) = 2\pi(1 - \mathrm{card}(\mathscr{F}_1) + \mathrm{card}(\mathscr{F}_2)).$$

This equality yields

$$\sum_{v \in F_0(\mathscr{P})} w_{\mathscr{P}}(z, v) = 2\pi(f_0(\mathscr{P}) - f_1(\mathscr{P}) + f_2(\mathscr{P})) = 2\pi\chi(\mathscr{P}),$$

which shows that $w_{\mathscr{P}}(z, v)$ is essentially a partitioning of the Euler-characteristic $\chi(\mathscr{P})$ of $\mathscr{P}$ on its vertices.

Let $\mathscr{B}$ be a K-realization associated with $\mathscr{P}$. The functional $w_{\mathscr{P},\mathscr{B}}(z, v)$ defined by

$$w_{\mathscr{P},\mathscr{B}}(z, v) := \begin{cases} w_{\mathscr{P}}(z, v) - w_{\mathscr{B}}(z, v) & \text{if } v \in F_0(\mathscr{B}) \\ w_{\mathscr{P}}(z, v) & \text{if } v \notin F_0(\mathscr{B}) \end{cases}$$

is called the index of v relative to $\mathscr{P}$ and $\mathscr{B}$ in direction z.

Obviously we have

$$\sum_{v \in F_0(\mathscr{P})} w_{\mathscr{P},\mathscr{B}}(z, v) = -4\pi g.$$

For a convex vertex v of $\mathscr{P}$, $w_{\mathscr{P}}(z, v)$ is non-negative. Since $w_{\mathscr{B}}(z, v) = 2\pi$ — this is equivalent to the fact that $H(z, v)$ supports $\mathrm{set}(\mathscr{B})$ at v — implies $w_{\mathscr{P}}(z, v) = 2\pi$, the index $w_{\mathscr{P},\mathscr{B}}(z, v)$ of a convex vertex is non-negative.

3. Bounds for the valence-functional

We are now going to prove that there is a "linear correspondence" between g and $V(g)$. The proof of the following theorem employs one lemma.

LEMMA. *Let $\mathscr{P}$ be a K-realization of (M, g), $\mathscr{B}$ a K-realization of $(M, 0)$ associated with $\mathscr{P}$ and v a non-convex 4-valent vertex of $\mathscr{P}$. Then there exists a direction z of $\mathrm{S}(\mathscr{P})$ such that $w_{\mathscr{P},\mathscr{B}}(z, v) \geqq 0$.*

PROOF. Let f be a facet of $\mathrm{st}(v, \mathscr{P})$, the angle at v of which is less than π and

z an element of $S(\mathcal{P})$, such that $f \subset H^+(z,v)$. Then $\mathrm{card}(\mathcal{G}(\mathcal{P};z,v)) \leqq 3$ and since this number is even we have $\mathrm{card}(\mathcal{G}(\mathcal{P};z,v)) \leqq 2$.

This implies $w_{\mathcal{P}}(z,v) = \pi(2\text{-}\mathrm{card}(\mathcal{G}(\mathcal{P};z,v))) \geqq 0$. Since $w_{\mathcal{B}}(z,v) = 2\pi$ implies $w_{\mathcal{P}}(z,v) = 2\pi$, we have $w_{\mathcal{P},\mathcal{B}}(z,v) \geqq 0$ which proves the lemma.

We give the upper bounds for $V(g)$ by constructing special K-realizations of (M,g) for every positive integer g where we need the well known process of cutting off a vertex. If $\mathcal{P}$ is a K-realization of (M,g) and E a set of convex vertices of $\mathcal{P}$ let us call any K-realization of (M,g) constructed in a successive process of cutting off the vertices of E E-truncation of $\mathcal{P}$.

THEOREM. *Let g be an integer with $g \geqq 1$. Then* $2g+1 \leqq V(g) \leqq 3g+3$.

This theorem shows the close relationship between g and $V(g)$. Though, it does not mean that the number of vertices which are more than 3-valent increases with increasing g. Surprisingly, this is not even true. Cutting off all convex vertices of the K-realization of (M,g) constructed in [3] to prove $H(g) \leqq 6$ shows the existence of a K-realization of (M,g), all but six vertices of which are 3-valent.

PROOF OF THE THEOREM. We start with the proof of the inequality from below. Let $\mathcal{P}$ be a K-realization of (M,g), $\mathcal{B}$ a K-realization of $(M,0)$ associated with $\mathcal{P}$ and $z \in S(\mathcal{P})$. For every non-negative integer j we define

$$w_j(\mathcal{P}) := \mathrm{card}(\{v \mid v \in F_0(\mathcal{P}) \wedge w_{\mathcal{P},\mathcal{B}}(z,v) = 2\pi(1-j)\}).$$

Since $\sum_{v \in F_0(\mathcal{P})} w_{\mathcal{P},\mathcal{B}}(z,v) = -4\pi g$, we get

$$\sum_{j \geqq 0} (j-1)\, w_j(\mathcal{P}) = 2g$$

and furthermore

$$\begin{aligned} v(\mathcal{P}) &\geqq \sum_{j \geqq 2} (2j-3)\, w_j(\mathcal{P}) \\ &= 2\sum_{j \geqq 2} (j-1)\, w_j(\mathcal{P}) - \sum_{j \geqq 2} w_j(\mathcal{P}) \\ &= 4g + 2\, w_0(\mathcal{P}) - \sum_{j \geqq 2} w_j(\mathcal{P}). \end{aligned}$$

In consequence of

$$\sum_{j \geqq 2} w_j(\mathcal{P}) \leqq 2g + w_0(\mathcal{P})$$

we get $v(\mathcal{P}) \geqq 2g + w_0(\mathcal{P})$, thus

$$v(\mathcal{P}) \geqq 2g.$$

In case of equality every non-convex vertex v is 4-valent and for each $z \in \mathbf{S}(\mathcal{P})$ we get $w_{\mathcal{P},\mathcal{B}}(z, v) = -2\pi$. This is a contradiction to the preceding lemma, which yields $2g + 1 \leqq V(g)$.

To prove the second inequality, for every positive integer g we construct a K-realization of (M, g) with valence-value $3g + 3$. For $g = 1$ it is very easy to construct such a K-realization starting with the well-known triangular picture frame, so we assume $g \geqq 2$.

The idea of the proof is to consider a 3-polytope in $\mathbb{R}^3$, construct another 3-polytope nicely situated inside the first one and take it away to generate the "holes" of the manifold. Let S denote a simplicial 3-polytope in $\mathbb{R}^3$ with $0 \in \operatorname{int}(S)$, $\mathcal{S}$ its boundary complex. We point out that S is homeomorphic to a 3-ball, whereas $\mathcal{S}$ is a K-realization of the sphere. Since the number of facets of a simplicial 3-polytope, which is closely related to the genus of the manifold we want to construct, is always even we have to distinguish between odd and even genera.

In the first case let g be odd and $f_0(\mathcal{S}) = \frac{1}{2}(g + 5)$. Then we get $f_1(\mathcal{S}) = \frac{3}{2}(g + 1)$ and $f_2(\mathcal{S}) = g + 1$. We now construct a polytope R_1 inside S by cutting off the edges of S. The following considerations are valid for a sufficiently small positive real ε. Let $\mathcal{S}_\varepsilon$ denote the boundary complex of the dilatation body $(1 - \varepsilon)S$ of S which will be denoted by S_ε. $\mathcal{S}_\varepsilon$ is isomorphic to $\mathcal{S}$ in a natural way and we mark the faces of $\mathcal{S}_\varepsilon$ corresponding to the faces of $\mathcal{S}$ they are associated with by the index ε. For every edge e_ε of $F_1(\mathcal{S}_\varepsilon)$ let H_{e_ε} denote a hyperplane supporting S_ε, such that $H_{e_\varepsilon} \cap S_\varepsilon = e_\varepsilon$ and $H^-_{e_\varepsilon}$ the associated halfspace not containing e. We can use these hyperplanes to cut off the edges of S and define $\bar{R}_1 := S \cap \bigcap_{e_\varepsilon \in F_1(\mathcal{S}_\varepsilon)} H^-_{e_\varepsilon}$.

Let $\bar{\mathcal{R}}_1$ denote the boundary complex of $\bar{R}_1$. Then $\bar{\mathcal{R}}_1$ is a K-realization of the sphere with $f_0(\bar{\mathcal{R}}_1) = f_0(\mathcal{S}) + 3f_2(\mathcal{S})$, all the vertices of $F_0(\bar{\mathcal{R}}_1) \setminus F_0(\mathcal{S}_\varepsilon)$ being 3-valent. In cutting off those vertices of $\bar{\mathcal{R}}_1$ which belong to $F_0(\mathcal{S}_\varepsilon)$, we obtain the boundary complex $\mathcal{R}_1$ of a simple polytope R_1 such that $F_0(\mathcal{R}_1) \setminus F_0(\bar{\mathcal{R}}_1) \subset \operatorname{int}(S)$.

We now construct a K-realization of (M, g). Define $\bar{P}_1 := \operatorname{bd}(S \setminus \operatorname{int}(R_1))$ and let $\bar{\mathcal{P}}_1$ denote the "natural" K-realization of $\bar{P}_1$, that is

$$\begin{aligned} F_1(\bar{\mathcal{P}}_1) = F_1(\mathcal{S}) \cup F_1(\mathcal{R}_1) \\ \cup \{e \mid e \subset \bar{P}_1 \wedge \exists v_1, v_2 : v_1 \in \bar{P}_1 \wedge v_2 \in F_0(\mathcal{S}) \wedge e = \operatorname{conv}(\{v_1, v_2\}) \\ \wedge \operatorname{conv}(\{v_1, v_{2,\varepsilon}\}) \in F_1(\bar{\mathcal{R}}_1)\}. \end{aligned}$$

One can easily see that $\bar{\mathscr{P}}_1$ is a K-realization of (M, g). To decrease the valence-value, we replace $\bar{\mathscr{P}}_1$ by its $F_0(\mathscr{S})$-truncation $\mathscr{P}_1$. All the vertices of $\mathscr{P}_1$ in $F_0(\mathscr{R}_1) \cap \mathrm{bd}(S)$ are 4-valent, whereas all the other vertices of $\mathscr{P}_1$ are 3-valent, thus $v(\mathscr{P}_1) = 3f_2(\mathscr{S}) = 3g + 3$ and $\mathscr{P}_1$ is the aspired K-realization in the case of odd g.

Figure 1 shows a K-realization $\mathscr{P}_1$ of $(M, 7)$ with $v(\mathscr{P}_1) = 24$.

Now we consider the case that g is even and suppose $f_0(\mathscr{S}) = \frac{1}{2}(g + 6)$. Let f be a facet of $\mathscr{S}$, H a hyperplane lying (strictly) between aff(f) and aff(f_ε) and let H^- denote the closed halfspace of H containing f_ε. Let $\bar{R}_2 := \bar{R}_1 \cap H^-$ and $\bar{\mathscr{R}}_2$ its boundary complex. The $F_0(\mathscr{S}_\varepsilon)$-truncation $\mathscr{R}_2$ of $\bar{\mathscr{R}}_2$ is again the boundary complex of a simple polytope. If we now construct $\mathscr{P}_2$ corresponding to the construction of $\mathscr{P}_1$, $\mathscr{P}_2$ has the aspired properties which completes the proof of the theorem.

It can be shown that for $g = 1,2$ the given lower bounds for $V(g)$ are not best possible. In fact, by means of extensive geometrical arguments it is proved in [3] that the lower bounds can be replaced by 6, which especially yields the exact value of $V(1)$.

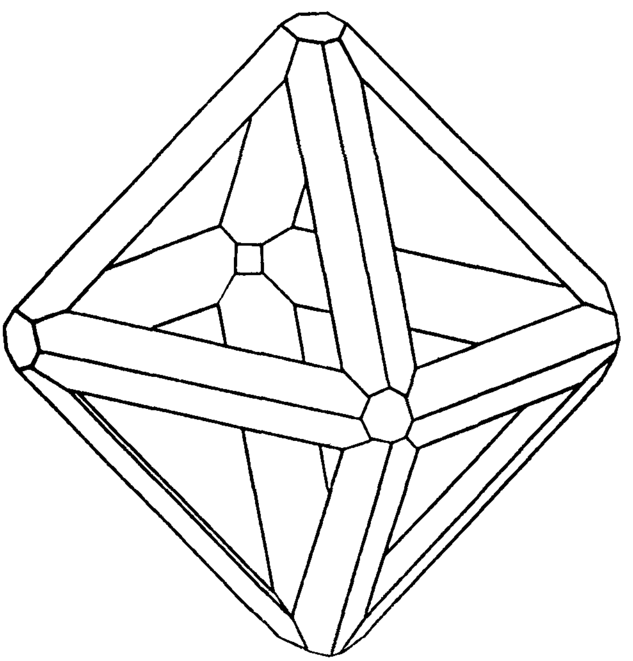

Fig. 1. K-realization $\mathscr{P}_1$ of $(M, 7)$ with $v(\mathscr{P}_1) = 24$.

4. Remarks

If $\mathcal{P}$ is a K-realization of (M,g) it is easily seen that the equality $f_2(\mathcal{P}) = \frac{1}{2}f_0(\mathcal{P}) + \frac{1}{2}v(\mathcal{P}) + 2(1-g)$ holds. This shows the importance of the valence-functional in connection with the determination of all possible f-vectors of K-realizations of (M,g). Indeed, the result $V(1)=6$ is sufficient to characterize the f-vectors of K-realizations of the torus (cf. [3], [4]).

The valence-functional is important for some other questions, too. If we ask, for instance, if K-realizations $\mathcal{P}$ of (M,g) exist such that for every integer j with $j \geqq 3$ the number $p_j(\mathcal{P})$ of j-gons of $\mathcal{P}$ equals a prescribed number, we are led to the problem of determining $V(g)$, too, because of the easy combinatorial equality

$$3p_3(\mathcal{P}) + 2p_4(\mathcal{P}) + p_5(\mathcal{P}) = 12(1-g) + 2v(\mathcal{P}) + \sum_{j \geqq 7} (j-6)p_j(\mathcal{P}).$$

Lower bounds for $V(g)$ yield necessary conditions for this problem, too.

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